MMP Learning Seminar

Week 71:

Introduction to dual complexes.

Centers & places:

(X,Δ) be a log canonical pair.

A log canonical place E of (X,Δ) is a divisor over X for which $a_E(X,\Delta)=0$.

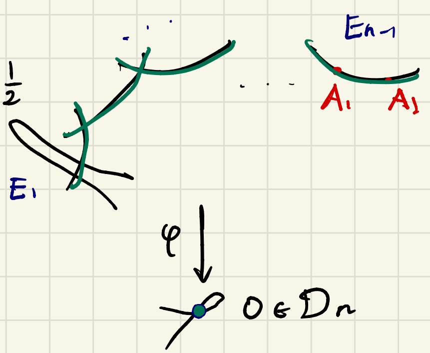
A log canonical center of (X,Δ) is the image $c_X(E)$ (or center) of a log canonical place

Proposition: Let (X,Δ) be a log canonical pair. The following statements hold:

i) (X,Δ) has finitely many lcc's.

ii) The intersection of two lcc's is union of lcc's.

Example: D_n singularity $\{x^2+y^2z+z^{n-1}=0\}$



$$\varphi^*(K_{X_{D_n}}+\tfrac{1}{2}c) = K_Y + E_1+\dots+E_{n-1}+\tfrac{1}{2}c$$

$(D_n, \frac{1}{2}c)$ strictly lc

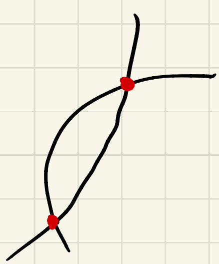
Divisorially log terminal sing (dlt sing):

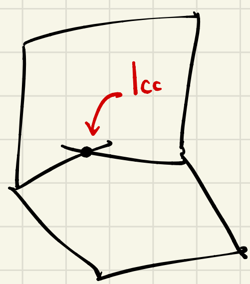
Definition: A log canonical pair (X, Δ) is said to be divisorially log terminal if there exists $U \subseteq X$:

1) U is smooth & $\lfloor \Delta \rfloor |_U$ is snc, and.
2) every lcc of (X, Δ) intersects U and is given by strata of $\lfloor \Delta \rfloor$.

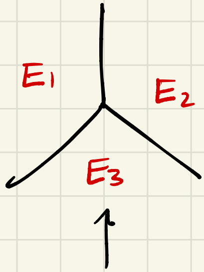
We say that a dlt pair (X, Δ) is purely log terminal if every lcc of (X, Δ) is divisorial.

Remark: (X, Δ) is dlt, then any connected intersection of comp of $\lfloor \Delta \rfloor$ is irreducible.

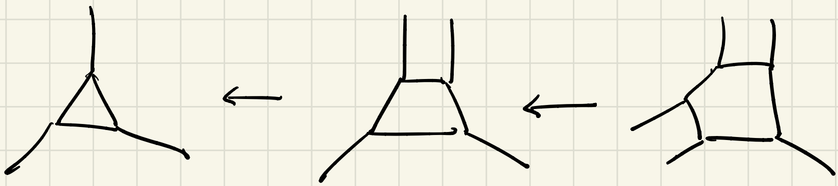


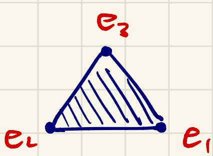


$$\begin{array}{ccccccccccc} & & A_n & & \text{quot surf not DV} & & (\mathbb{A}^2, L) & & (\mathbb{A}^2, L_1+L_2) & & \text{cone over elliptic} \\ \text{terminal} & \Rightarrow & \text{canonical} & \Rightarrow & \text{klt} & \Rightarrow & \text{plt} & \Rightarrow & \text{dlt} & \Rightarrow & \text{lc.} \end{array}$$



$(\mathbb{A}^3, H_1 + H_2 + H_3)$





n $\quad H_1 + \cdots + H_n$

log canonical places of $(\mathbb{A}^3, H_1 + H_2 + H_3)$

are parametrized by $\mathbb{Q}$-points in the

$(n-1)$ - simplex.

standard 2- simplex.

Proposition: Let (X, Δ) be a log canonical pair.

There exists a projective birational morphism $\pi : Y \longrightarrow X$:

i) the exceptional locus $\mathrm{Ex}(\pi)$ is purely div.

ii) every prime $E \subseteq \mathrm{Ex}(\pi)$ satisfies $a_E(X, \Delta) = 0$, and.

iii) the pair (Y, Δ_Y) is dlt where $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$

Remark 1: (T, Δ_T) reduced toric pair, then a dlt

modification is a log resolution

Rmk 2: In dim ≥ 3, there is no minimal dlt mod.

Dual complexes:

$Z = \bigcup_{i \in I} Z_i$ pure dim scheme with comp Z_i. Assume:

i) each comp Z_i is normal, and

ii) for every $J \subseteq I$, if $\bigcap_{i \in J} Z_j$ is non-empty.

then every comp of $\bigcap_{i \in J} Z_j$ is irreducible.

dual complex of Z

$D(Z)$ is a regular complex cell obtained as following:

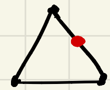
$$\text{vertices} \longleftrightarrow Z_i.$$

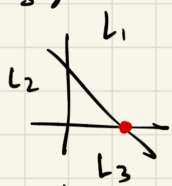
each comp $W \subseteq \bigcap_{j \in J} Z_j \longleftrightarrow v_W$ of dim $|J|-1$.

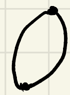
Def 1: (Y, Δ_Y) is dlt $\quad D(Y, \Delta_Y) = D(\lfloor \Delta_Y \rfloor)$

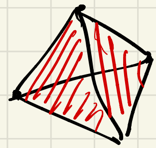
Def 2: The dual complex of (X, Δ) associated to the dlt modification $(Y, \Delta_Y) \longrightarrow (X, \Delta)$ is denoted by $D(Y, \Delta_Y) := D(\lfloor \Delta_Y \rfloor)$.

Examples & Theorems:

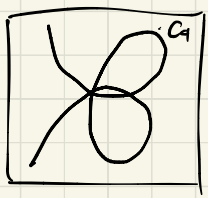
$(\mathbb{P}^2, L_1 + L_2 + L_3)$ $\mathcal{D} =$ 

$(\mathbb{P}^2, C + L)$ 

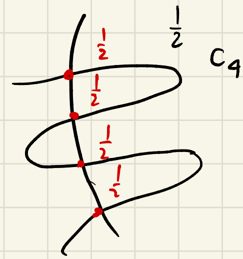
$\mathcal{D} =$ 

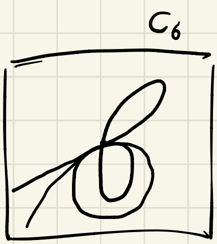
$(\mathbb{P}^3, H_1 + \dots + H_4)$ $\mathcal{D} =$ 

blow-ups of strata of dlt pairs replace $D(X, \Delta)$ with a barycenter subdivision



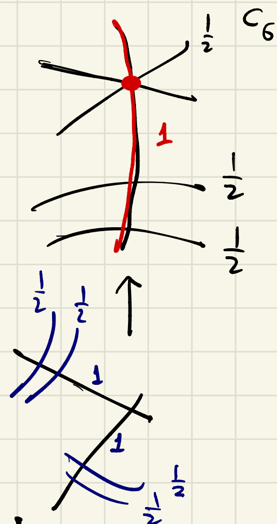
log canonical $(\mathbb{P}^2, \frac{1}{2} C_4)$ $\mathcal{D} = \cdot$



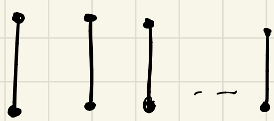


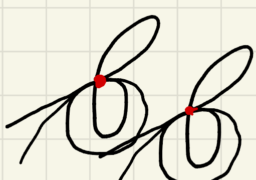
log canonical $(\mathbb{P}^2, \frac{1}{2} C_6)$

$\mathcal{D} =$ 



$\frac{1}{2} C_6'' + \dots$

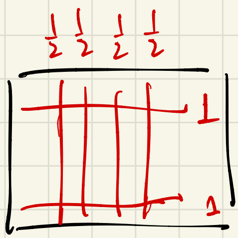
$\mathcal{D}(\mathbb{P}^2, \frac{1}{2} C_6 + \frac{1}{2} C_6') =$ 



Theorem: Let (X,Δ) be a dlt pair of dim n & $-(K_X+\Delta)$ is ample.
Then $\mathcal{D}(X,\Delta) \simeq S_k$ where $k \leq n-1$.

Theorem: Let (X,Δ) be a log canonical pair which is log CY, $K_X+\Delta \equiv 0$. Then the set of log canonical centers is connected unless the dlt modification of (X,Δ) is plt and has exactly two disjoint components.

Example:



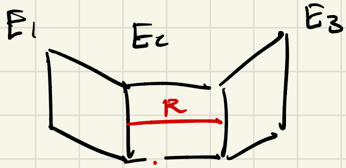
$\mathbb{P}^1 \times \mathbb{P}^1$.

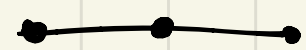
$$\overset{\text{dlt}}{(X,\Delta)} \dashrightarrow \overset{\text{dlt}}{(Y,\Delta_Y)}$$

$(K_X+\Delta)$-MMP

Q: How do $\mathcal{D}(X,\Delta)$ and $\mathcal{D}(Y,\Delta_Y)$ compare?

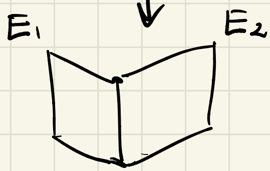
(X,Δ)



$\mathcal{D} =$ 

$\downarrow$ crepant.

(Y,Δ_Y)



$\mathcal{D} =$ 

Dual complexes under the MMP:

Theorem: Let (X, Δ) be a dlt pair. & $f: X \dashrightarrow Y$ a $(K_X + \Delta)$-MMP step with extremal ray R. Set $\Delta_Y = f_* \Delta$. Assume there is $D_0 \subseteq \lfloor \Delta \rfloor$ a component with $D_0 \cdot R > 0$. Then $\mathcal{D}(X, \Delta)$ collapses to $\mathcal{D}(Y, \Delta_Y)$.

Sketch: $Z \subseteq Ex(f)$ and Z is an irreducible comp of $D_0 \cap \bigcap_{i \in J} D_i$

Then, there exists Z^+ comp of $\bigcap_{i \in J} D_i$ with $Z^+ \cap D_0 = Z$, $Z^+ \subseteq Ex(f)$.

$W \not\subseteq D_0$ with $Ex(f) \supseteq W$, then $W^- := W \cap D_0$ is $Ex(f) \supseteq W^-$.

$$Z \longmapsto Z^+ \qquad W \longmapsto W^-$$

$M \subseteq D(X, \Delta)$ union of all cells

v_Z where $Z \subseteq Ex(f)$

$$D(Y, \Delta_Y) = D(X, \Delta) \setminus M.$$

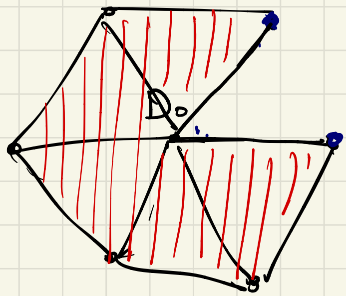
cells of M come in pairs.

$$(\langle v_{D_0}, v_W \rangle, v_W).$$

$$\langle v_{D_0}, v_W \rangle = v_{D_0 \cap W}$$

star type

Link type



If $v_W \in M$ is a facet of v_V, then $v_V \subseteq M$.

If $(\langle v_0, v_W \rangle, v_W)$ is maximal, then is a free pair.

$\square$

Corollary 22: (X,Δ) dlt, $g: X \to Z$

$f: X \dashrightarrow Y$ $(K_X+\Delta)$-MMP step over Z.

$\Delta_Y = f_* \Delta$. Assume there is a g-trivial effective divisor whose suppor is $\lfloor \Delta \rfloor$.

Then $\mathcal{D}(X,\Delta)$ collapses to $\mathcal{D}(\Delta_Y^{=1})$.

Lemma 23: X, Y $\mathbb{Q}$-factorial normal varieties.

$p: Y \to X$ projective birational. $\Gamma \subseteq Y$ s.t (Y,Γ) is lc

Let $f: Y \dashrightarrow Y_1$ a $(K_Y+\Gamma)$-MMP step. over X.

Let R be the corresponding $(K_Y+\Gamma)$-extremal neg ray. Then:

i) There exists $E_R \subseteq Ex(p)$ s.t $(E_R \cdot R) > 0$, or

ii) f contracts $E_f \subseteq Ex(p)$ and $Y_1 \to X$ is an isom at the generic point of $f(E_f)$.

Theorem 3: (X,Δ) quasi-proj dlt.

$g: Y \longrightarrow X$ log resolution.

$E := \operatorname{Supp} g^{-1}(\lfloor \Delta \rfloor)$.

Assume X is $\mathbb{Q}$-factorial.

Then $\mathcal{D}(E)$ collapses to $\mathcal{D}(X,\Delta)$.

Proof: $\Gamma = g_*^{-1}\Delta^{<1} + E + \sum_i \frac{1}{2} F_i$.

$\mathbf{B}_{-}(K_Y + \Gamma) \supseteq F_i$ for each i.

$(K_Y+\Gamma)$-MMP over X

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d] & (Y', \Delta_{Y'}) \arrow[dl] \\ (X,\Delta) & \end{tikzcd}$$

$$\begin{tikzcd} & \mathcal{D}(Y,\Gamma) = \mathcal{D}(E) \arrow[d, "\text{collapses}"] \\ \mathcal{D}(X,\Delta) & \mathcal{D}(Y', \Delta_{Y'}) \arrow[l, "\text{collapse}"'] \end{tikzcd}$$

$\square$

Corollary: (X, Δ) log canonical.

Then the PL-homeomorphism class of $\mathcal{D}(X, \Delta)$ is well-defined.